# Magnetic moment of the muon, the search for new fundamental physics and a lattice QCD calculation of hadronic vacuum polarization

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Budapest-Marseille-Wuppertal collaboration [BMWc]

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Nature 593 (2021) 51, online 7 April 2021 → BMWc '20
PRL 121 (2018) 022002 (Editors' Selection) → BMWc '17
& Aoyama et al., Phys. Rept. 887 (2020) 1-166 → WP '20

# Lepton magnetic moments and BSM physics

# Interaction with an external EM field: QM

Dirac eq. w/ minimal coupling (1928):

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\vec{\alpha}\cdot\left(c\frac{\hbar}{i}\vec{\nabla} - e_\ell\vec{A}\right) + \beta c^2 m_\ell + e_\ell A_0\right]\psi$$

nonrelativistic limit $\downarrow$ (Pauli eq.)

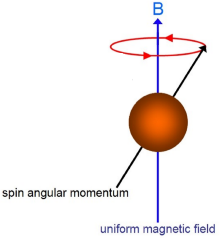
$$i\hbar\frac{\partial\phi}{\partial t} = \left[\frac{\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e_\ell}{c}\vec{A}\right)^2}{2m_\ell} - \underbrace{\frac{e_\ell\hbar}{2m_\ell}\vec{\sigma}\cdot\vec{B}}_{\vec{\mu}_\ell\cdot\vec{B}} + e_\ell A_0\right]\phi$$

with

$$\vec{\mu}_\ell = g_\ell\left(\frac{e_\ell}{2m_\ell}\right)\vec{S}, \qquad \vec{S} = \hbar\frac{\vec{\sigma}}{2}$$

and

$$g_\ell\big|_{\text{Dirac}} = 2$$




*"That was really an unexpected bonus for me, completely unexpected." (P.A.M. Dirac)*

# Interaction with an external EM field: QFT

Assuming Poincaré invariance and current conservation ($q^\mu J_\mu = 0$ with $q \equiv p' - p$):

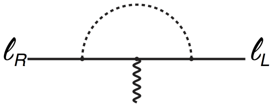
$$\langle \ell(p')|J_\mu(0)|\ell(p)\rangle = \bar{u}(p')\left[\gamma_\mu F_1(q^2) + \frac{i}{2m_\ell}\sigma_{\mu\nu}q^\nu F_2(q^2) - \gamma_5\sigma_{\mu\nu}q^\nu F_3(q^2)\right.$$

$$\left. +\gamma_5(q^2\gamma_\mu - 2m_\ell q_\mu)F_4(q^2)\right] u(p)$$

| | | |
|---|---|---|
| $F_1(q^2)$ | $\rightarrow$ | Dirac form factor: $F_1(0) = 1$ |
| $F_2(q^2)$ | $\rightarrow$ | Pauli form factor, magnetic dipole moment: $F_2(0) = a_\ell = \frac{g_\ell - \overbrace{2}^{g_\ell\vert_{\text{Dirac}}}}{2}$ |
| $F_3(q^2)$ | $\rightarrow$ | $\not{P}$, $\not{T}$, electric dipole moment: $F_3(0) = d_\ell / e_\ell$ |
| $F_4(q^2)$ | $\rightarrow$ | $\not{P}$, anapole moment: $\vec{\sigma} \cdot (\vec{\nabla} \times \vec{B})$ |

- $q^2$ dependence of $F_1(q^2)$ and non-zero $F_2(q^2)$ & $F_{3,4}(q^2)$ come from loops but UV finite once charges and masses are renormalized (in a renormalizable theory)
- $a_\ell$ dimensionless & $m_\gamma = 0$
  - $\Rightarrow$ corrections including only $\ell$ and $\gamma$ are mass independent, i.e. universal
  - $\rightarrow$ contributions from particles w/ $M \gg m_\ell$ are $\propto (m_\ell/M)^{2p} \times \ln^q(m_\ell^2/M^2)$
  - $\rightarrow$ contributions from pions w/ $m_\pi \ll m_\ell$ are e.g. $\propto \ln(m_\ell^2/m_\pi^2)$

# Why are $a_\ell$ special?




$$\longrightarrow \quad \frac{a_\ell}{2m_\ell} eF^{\mu\nu}[\bar{\ell}_L \sigma_{\mu\nu} \ell_R]$$

- $a_{e,\mu}$ are parameter-free predictions of the SM that can be measured very precisely $\Rightarrow$ excellent tests of SM
- Loop induced $\Rightarrow$ sensitive to new dofs that may be too heavy or too weakly coupled to be produced directly
- CP and flavor conserving, chirality flipping $\Rightarrow$ complementary to: EDMs, $s$ and $b$ decays, LHC direct searches, ...
- Chirality flipping $\Rightarrow$ generic contribution of particle w/ $M \gg m_\ell$

$$a_\ell^{\mathrm{M}} = C \left(\frac{\Delta_{LR}}{m_\ell}\right) \left(\frac{m_\ell}{M}\right)^2$$

- In EW theory, $M = M_W$, chirality flipping from Yukawa, i.e.

$$\Delta_{LR} = m_\ell \qquad \text{and} \qquad C \sim \frac{\alpha}{4\pi \sin^2 \theta_W}$$

- In BSM, can have chiral enhancement: e.g. SUSY $M = M_{\mathrm{SUSY}}$ and $C \sim \alpha/\left(4\pi \sin^2 \theta_W\right)$ & $\Delta_{LR} = \left(\mu/M_{\mathrm{SUSY}}\right) \times \tan\beta \times m_\ell$; or radiative $m_\ell$ model, $\Delta_{LR} \simeq m_\ell$, $C \sim 1$ and $M = M_{\mathrm{N}\Phi}$

# Why is $a_\mu$ special?

$m_e : m_\mu : m_\tau = 0.0005 : 0.106 : 1.777\,\mathrm{GeV}$ $\qquad \tau_e : \tau_\mu : \tau_\tau =$ "$\infty$" $: 2.\cdot10^{-6} : 3.\cdot10^{-15}$ s

- $a_\mu$ is $(m_\mu/m_e)^2 \sim 4. \times 10^4$ times more sensitive to new Φ than $a_e$
- $a_\tau$ is even more sensitive to new Φ, but is too shortly lived
- $\tau_\mu$ small but manageable

$\rightarrow$ measure & compute $a_\mu$ in SM as precisely as possible

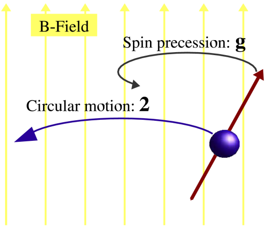
Big question:

$$a_\mu^{\text{exp}} = a_\mu^{\text{SM}}?$$

If not, there must be new Φ

# Experimental measurement of $a_\mu$

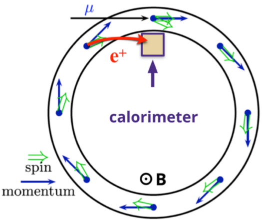
# Measurement principle for $a_\mu$




Precession determined by

$$\vec{\mu}_\mu = 2(1 + a_\mu)\frac{Qe}{2m_\mu}\vec{S}$$

$$\vec{d}_\mu = \eta_\mu \frac{Qe}{2m_\mu c}\vec{S}$$




$$\vec{\omega}_{a\eta} = \vec{\omega}_a + \vec{\omega}_\eta = -\frac{Qe}{m_\mu}\left[a_\mu \vec{B} + \left(a_\mu - \frac{1}{\gamma^2 - 1}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right] - \eta_\mu \frac{Qe}{2m_\mu}\left[\frac{\vec{E}}{c} + \vec{\beta}\times\vec{B}\right]$$

- Experiment measures very precisely $\vec{B}$ with $|\vec{B}| \gg |\vec{E}|/c$ &

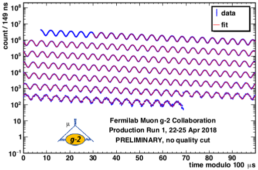
$$\Delta\omega \equiv \omega_S - \omega_C \simeq \sqrt{\omega_a^2 + \omega_\eta^2} \simeq \omega_a$$

since $d_\mu = 0.1(9) \times 10^{-19} e \cdot \mathrm{cm}$ (Benett et al '09)

- Consider either magic $\gamma = 29.3$ (CERN/BNL/Fermilab) or $\vec{E} = 0$ (J-PARC)

$$\to \Delta\omega \simeq -a_\mu B \frac{Qe}{m_\mu}$$

# Fermilab E989 @ magic $\gamma$: measurement (simplified)



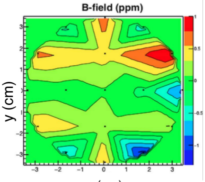

$\omega_a$

Extract from decay positron time spectra

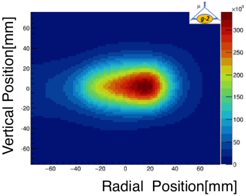
$$N(t) = N_0 e^{-t/\tau_\mu}[1 + A\cos(\omega_a t + \phi)]$$

$$a_\mu = \left(\frac{g_e}{2}\right)\left(\frac{\omega_a}{\langle\omega_p\rangle}\right)\left(\frac{\mu_p}{\mu_e}\right)\left(\frac{m_\mu}{m_e}\right)$$

0.26 ppt — 3 ppb — 22 ppb ⇨ 2017 CODATA




Map the magnetic field




Obtain muon distribution In the storage ring

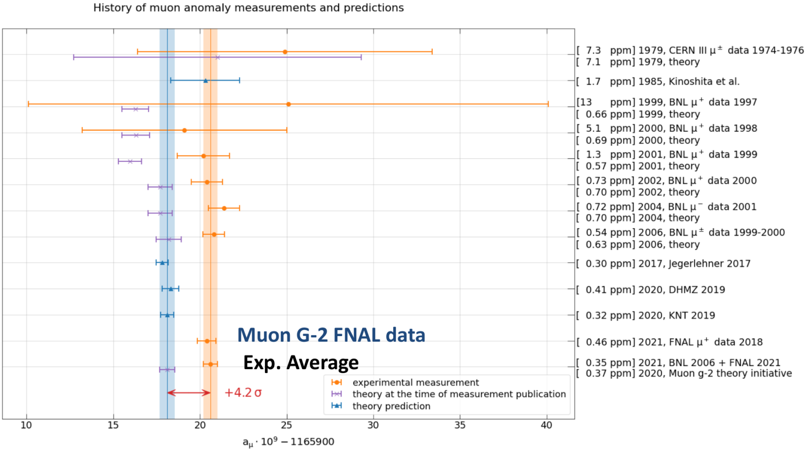
$$\langle\omega_p\rangle \approx \omega_p \otimes \rho(r)$$

Average magnetic field weighted by muon distribution

$\omega_p$: free proton precession frequency

Using proton NMR $\hbar\omega_p = 2\mu_p B$

# $g_\mu - 2$ updated history (7 April 2021)




$$a_\mu(\text{AVG}) = 116\,592\,061(41) \times 10^{-11} \quad (0.35\,\text{ppm}).$$

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G. Venanzoni, CERN Seminar, 8 April 2021

Bathroom scale sensitive to the weight of a single eyelash !!!

Based on only 6% of expected FNAL data! $\rightarrow$ aim $\delta a_\mu = 0.14$ ppm

# Standard model calculation of $a_\mu$

At needed precision: all three interactions and all SM particles

$$
\begin{aligned}
a_\mu^{\text{SM}} &= a_\mu^{\text{QED}} + a_\mu^{\text{had}} + a_\mu^{\text{EW}} \\
&= O\left(\frac{\alpha}{2\pi}\right) + O\left(\left(\frac{\alpha}{\pi}\right)^2\left(\frac{m_\mu}{M_\rho}\right)^2\right) + O\left(\left(\frac{e}{4\pi\sin\theta_W}\right)^2\left(\frac{m_\mu}{M_W}\right)^2\right) \\
&= O\left(10^{-3}\right) + O\left(10^{-7}\right) + O\left(10^{-9}\right)
\end{aligned}
$$

# QED contributions to $a_\ell$

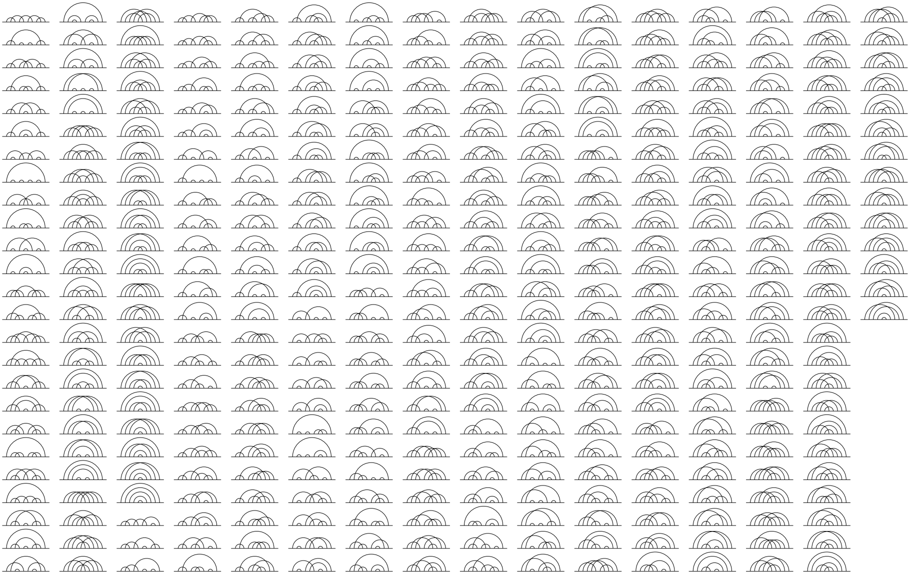
Loops with only photons and leptons

$$a_\ell^{\text{QED}} = C_\ell^{(2)}\left(\frac{\alpha}{\pi}\right) + C_\ell^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + C_\ell^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + C_\ell^{(8)}\left(\frac{\alpha}{\pi}\right)^4 + C_\ell^{(10)}\left(\frac{\alpha}{\pi}\right)^5 + \cdots$$

$$C_\ell^{(2n)} = A_1^{(2n)} + A_2^{(2n)}(m_\ell/m_{\ell'}) + A_3^{(2n)}(m_\ell/m_{\ell'}, m_\ell/m_{\ell''})$$

- $A_1^{(2)}, A_1^{(4)}, A_1^{(6)}, A_2^{(4)}, A_2^{(6)}, A_3^{(6)}$ known analytically (Schwinger '48; Sommerfield '57, '58; Petermann '57; ...)
- $O((\alpha/\pi)^3)$: 72 diagrams (Laporta et al '91, '93, '95, '96; Kinoshita '95)
- $O((\alpha/\pi)^4; (\alpha/\pi)^5)$: 891;12,672 diagrams (Laporta '95; Aguilar et al '08; Aoyama, Kinoshita, Nio '96-'18)
  - Automated generation of diagrams
  - Numerical evaluation of loop integrals
  - Not all contributions are fully, independently checked

# 5-loop QED diagrams



(Aoyama et al '15)

# QED contribution to $a_\mu$

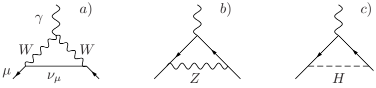
$$a_\mu^{\rm QED}(Cs) = 1\,165\,847\,189.31(7)_{m_\tau}(17)_{\alpha^4}(6)_{\alpha^5}(100)_{\alpha^6}(23)_{\alpha(Cs)} \times 10^{-12}\ [0.9\ \text{ppb}]$$
$$a_\mu^{\rm QED}(a_e) = 1\,165\,847\,188.42(7)_{m_\tau}(17)_{\alpha^4}(6)_{\alpha^5}(100)_{\alpha^6}(28)_{\alpha(a_e)} \times 10^{-12}[0.9\ \text{ppb}]$$

(Aoyama et al '12, '18, '19)

$$a_\mu^{\rm exp} - a_\mu^{\rm QED} = 734.2(4.1) \times 10^{-10}$$
$$\stackrel{?}{=} a_\mu^{\rm EW} + a_\mu^{\rm had}$$

# Electroweak contributions to $a_\mu$: $Z$, $W$, $H$, etc. loops

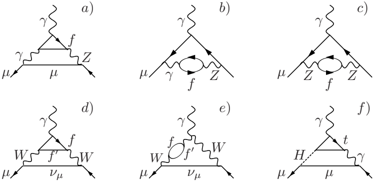
**1-loop**




$$a_\mu^{\text{EW},(1)} = O\left(\frac{\sqrt{2}G_F m_\mu^2}{16\pi^2}\right) = 19.479(1)\times 10^{-10}$$

(Gnendiger et al '15, Aoyama et al '20 and refs therein)

**2-loop**




$$a_\mu^{\text{EW},(2)} = O\left(\frac{\sqrt{2}G_F m_\mu^2}{16\pi^2}\frac{\alpha}{\pi}\right) = -4.12(10)\times 10^{-10}$$

(Gnendiger et al '15 and refs therein)

$$a_\mu^{\text{EW}} = 15.36(10)\times 10^{-10}$$

# Hadronic contributions to $a_\mu$: quark and gluon loops

$$a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{EW}} = 718.9(4.1) \times 10^{-10} \stackrel{?}{=} a_\mu^{\text{had}}$$

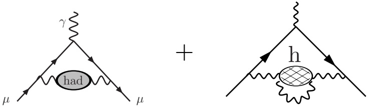
Clearly right order of magnitude:

$$a_\mu^{\text{had}} = O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) = O\left(10^{-7}\right)$$

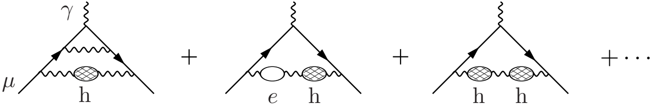
(already Gourdin & de Rafael '69 found $a_\mu^{\text{had}} = 650(50) \times 10^{-10}$)

Write

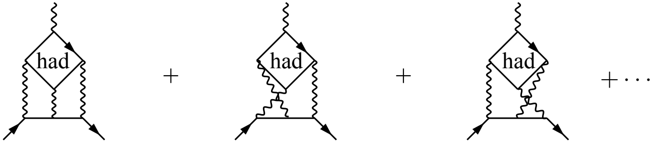
$$a_\mu^{\text{had}} = a_\mu^{\text{LO-HVP}} + a_\mu^{\text{HO-HVP}} + a_\mu^{\text{HLbyL}} + O\left(\left(\frac{\alpha}{\pi}\right)^4\right)$$




$\rightarrow a_\mu^{\text{LO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^2\right)$

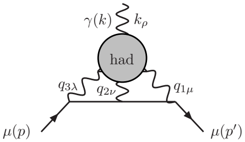



$\rightarrow a_\mu^{\text{NLO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$

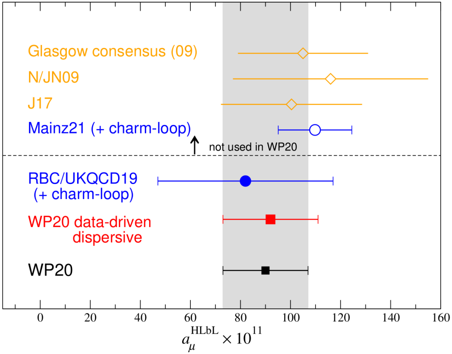



$\rightarrow a_\mu^{\text{HLbL}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$

# Hadronic light-by-light

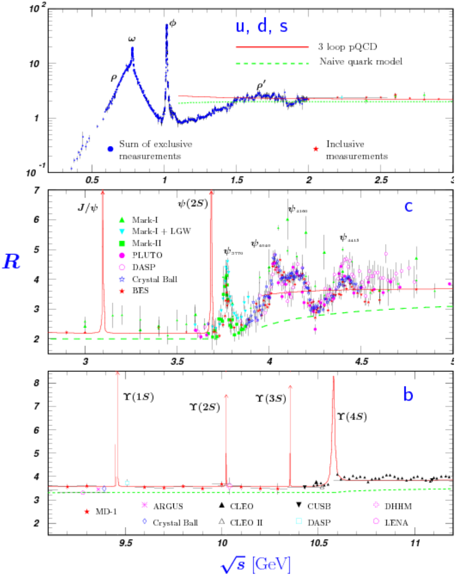



- HLbL much more complicated than HVP, but ultimate precision needed is $\simeq 10\%$ instead of $\simeq 0.2\%$
- For many years, only accessible to models of QCD w/ difficult to estimate systematics (Prades et al '09): $a_\mu^{\text{HLbL}} = 10.5(2.6) \times 10^{-10}$
- Also, lattice QCD calculations were exploratory and incomplete
- Tremendous progress in past 5 years:
  - → Phenomenology: rigorous data driven approach [Colangelo, Hoferichter, Kubis, Procura, Stoffer,... '15-'20]
  - → Lattice: first two solid lattice calculations
- All agree w/ older model results but error estimate much more solid and will improve
- Agreed upon average w/ NLO HLbL and conservative error estimates [WP '20]
- $a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{EW}} - a_\mu^{\text{HLbL}} = 709.7(4.5) \times 10^{-10} \stackrel{?}{=} a_\mu^{\text{HVP}}$



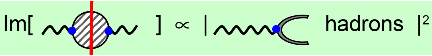

[Colangelo '21]

# HVP from $e^+e^- \to$ had (or $\tau \to \nu_\tau +$ had)




(PDG compilation)

Use [Bouchiat et al 61] optical theorem (unitarity)




$$\mathrm{Im}\Pi(s) = -\frac{R(s)}{12\pi}, \quad R(s) \equiv \frac{\sigma(e^+e^- \to \text{had})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

and a once subtracted dispersion relation (analyticity)

$$\hat{\Pi}(Q^2) = \int_0^\infty ds \, \frac{Q^2}{s(s+Q^2)} \frac{1}{\pi} \mathrm{Im}\Pi(s) = \frac{Q^2}{12\pi^2} \int_0^\infty ds \, \frac{1}{s(s+Q^2)} R(s)$$

$\Rightarrow \hat{\Pi}(Q^2)$ & $a_\mu^{\text{LO-HVP}}$ from data: sum of exclusive $\pi^+\pi^-$ etc. channels from CMD-2&3, SND, BES, KLOE '08,'10&'12, BABAR '09, etc.

$$a_\mu^{\text{LO-HVP}} = 694.0(1.0)(3.9) \times 10^{-10} \; [0.6\%]$$

[DHMZ'19] (sys. domin.)

Can also use $I(J^{PC}) = 1(1^{--})$ part of $\tau \to \nu_\tau +$ had and isospin symmetry + corrections

# Standard model prediction and comparison to experiment

# SM vs experiment: 7 April 2021, official version

| SM contribution | $a_\mu^{\text{contrib.}} \times 10^{10}$ | Ref. |
|---|---|---|
| HVP LO (R-ratio) | $692.8 \pm 2.4$ | [KNT '19] |
| | $694.0 \pm 4.0$ | [DHMZ '19] |
| | $692.3 \pm 3.3$ | [CHHKS '19] |
| HVP LO (R-ratio, avg) | $693.1 \pm 4.0$ | [WP '20] |
| HVP LO (lattice$<$2021) | $711.6 \pm 18.4$ | [WP '20] |
| HVP NLO | $-9.83 \pm 0.07$ | |
| | | [Kurz et al '14, Jegerlehner '16, WP '20] |
| HVP NNLO | $1.24 \pm 0.01$ | [Kurz '14, Jeger. '16] |
| HLbyL LO (pheno) | $9.2 \pm 1.9$ | [WP '20] |
| HLbyL LO (lattice$<$2021) | $7.8 \pm 3.1 \pm 1.8$ | [RBC '19] |
| HLbyL LO (lattice 2021) | $10.7 \pm 1.1 \pm 0.9$ | [Mainz '21] |
| HLbyL LO (avg) | $9.0 \pm 1.7$ | [WP '20] |
| HLbyL NLO (pheno) | $0.2 \pm 0.1$ | [WP '20] |
| QED [5 loops] | $11658471.8931 \pm 0.0104$ | [Aoyama '19, WP '20] |
| EW [2 loops] | $15.36 \pm 0.10$ | [Gnendiger '15, WP '20] |
| HVP Tot. (R-ratio) | $684.5 \pm 4.0$ | [WP '20] |
| HLbL Tot. | $9.2 \pm 1.8$ | [WP '20] |
| SM [0.37 ppm] | $11659181.0 \pm 4.3$ | [WP '20] |
| Exp [0.35 ppm] | $11659206.1 \pm 4.1$ | [BNL '06 + FNAL '21] |
| Exp $-$ SM | $25.1 \pm 5.9$ [$4.2\sigma$] | |

# SM vs experiment: 7 April 2021, alternative

| SM contribution | $a_\mu^{\text{contrib.}} \times 10^{10}$ | Ref. |
|---|---|---|
| HVP LO (R-ratio) | $692.8 \pm 2.4$ | [KNT '19] |
| | $694.0 \pm 4.0$ | [DHMZ '19] |
| | $692.3 \pm 3.3$ | [CHHKS '19] |
| HVP LO (R-ratio, avg) | $693.1 \pm 4.0$ | [WP '20] |
| HVP LO (lattice) | $707.5 \pm 5.5$ | [BMWc '20] |
| HVP NLO | $-9.83 \pm 0.07$ | |
| | | [Kurz et al '14, Jegerlehner '16, WP '20] |
| HVP NNLO | $1.24 \pm 0.01$ | [Kurz '14, Jeger. '16] |
| HLbyL LO (pheno) | $9.2 \pm 1.9$ | [WP '20] |
| HLbyL LO (lattice<2021) | $7.8 \pm 3.1 \pm 1.8$ | [RBC '19] |
| HLbyL LO (lattice 2021) | $10.7 \pm 1.1 \pm 0.9$ | [Mainz '21] |
| HLbyL LO (avg) | $9.0 \pm 1.7$ | [WP '20] |
| HLbyL NLO (pheno) | $0.2 \pm 0.1$ | [WP '20] |
| QED [5 loops] | $11658471.8931 \pm 0.0104$ | [Aoyama '19, WP '20] |
| EW [2 loops] | $15.36 \pm 0.10$ | [Gnendiger '15, WP '20] |
| HVP Tot. (lat. + R-ratio) | $698.9 \pm 5.5$ | [WP '20, BMWc '20] |
| HLbL Tot. | $9.2 \pm 1.8$ | [WP '20] |
| SM [0.49 ppm] | $11659195.4 \pm 5.7$ | [WP '20 + BMWc '20] |
| Exp [0.35 ppm] | $11659206.1 \pm 4.1$ | [BNL '06 + FNAL '21] |
| Exp − SM | $10.7 \pm 7.0$ [$1.5\sigma$] | |

# Very brief introduction to lattice QCD

# What is lattice QCD (LQCD)?

To describe matter w/ sub-% precision, QCD requires $\geq$ 104 numbers at every spacetime point

$\rightarrow \infty$ number of numbers in our continuous spacetime

$\rightarrow$ must temporarily "simplify" the theory to be able to calculate *(regularization)*

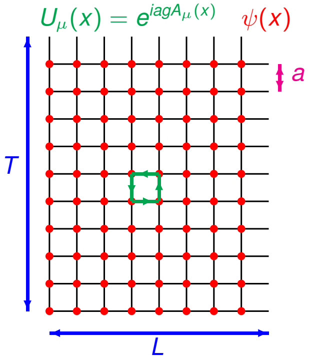
$\Rightarrow$ Lattice gauge theory $\longrightarrow$ mathematically sound definition of NP QCD:

- UV (& IR) cutoff $\rightarrow$ well defined path integral in Euclidean spacetime:

$$\langle O \rangle = \int \mathcal{D}U\mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-S_G - \int \bar{\psi}D[M]\psi}\, O[U,\psi,\bar{\psi}]$$

$$= \int \mathcal{D}U\, e^{-S_G}\,\det(D[M])\, O[U]_{\mathrm{Wick}}$$

- $\mathcal{D}Ue^{-S_G}\det(D[M]) \geq 0$ & finite # of dofs $\rightarrow$ evaluate numerically using stochastic methods




LQCD is QCD when $m_q \rightarrow m_q^{\mathrm{ph}}$, $a \rightarrow 0$ (after renormalization), $L \rightarrow \infty$ (and stats $\rightarrow \infty$)

HUGE conceptual and numerical ($O(10^9)$ dofs) challenge

# Our “accelerators”

Such computations require some of the world’s most powerful supercomputers




- $1$ year on supercomputer
  $\sim$ $100\,000$ years on laptop

- In Germany, those of the Forschungszentrum Jülich, the Leibniz Supercomputing Centre (Munich), and the High Performance Computing Center (Stuttgart); in France, Turing and Jean Zay at the Institute for Development and Resources in Intensive Scientific Computing (IDRIS) of the CNRS, and Joliot-Curie at the Very Large Computing Centre (TGCC) of the CEA, by way of the French Large-scale Computing Infrastructure (GENCI).

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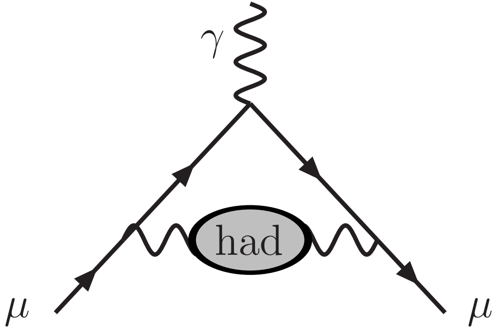
# Lattice QCD calculation of $a_\mu^{\text{HVP}}$




All quantities related to $a_\mu$ will be given in units of $10^{-10}$

# HVP from LQCD: introduction

Consider in Euclidean spacetime, i.e. spacelike $q^2 = -Q^2 \leq 0$ [Blum '02]

$$\Pi_{\mu\nu}(Q) =$$


$$= \int d^4x\, e^{iQ\cdot x}\langle J_\mu(x)J_\nu(0)\rangle$$

$$= \left(Q_\mu Q_\nu - \delta_{\mu\nu}Q^2\right)\Pi(Q^2)$$

w/ $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots$

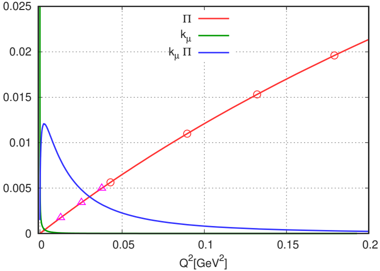
Then [Lautrup et al '69, Blum '02]

$$a_\ell^{\text{LO-HVP}} = \alpha^2 \int_0^\infty \frac{dQ^2}{m_\ell^2}\, k(Q^2/m_\ell^2)\hat{\Pi}(Q^2)$$

w/ $\hat{\Pi}(Q^2) \equiv \left[\Pi(Q^2) - \Pi(0)\right]$

Huge challenge: important long-distance contributions and sub-% precision needed
→ physically light $u$ & $d$ quarks, very large volumes, accurate $Q^2 = 0$ subtraction, many contributions, very high statistics, ...

Contributions of $ud$, $s$, $c$ ... have very different systematics (and statistical errors) on lattice
→ study each one individually




$(k_\mu(Q^2) = (\pi/m_\mu)^2 k(Q^2))$

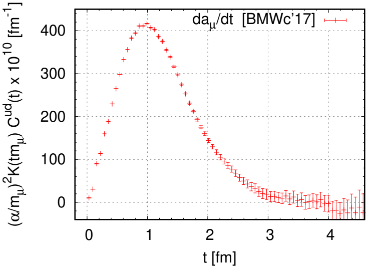
# Our lattice definition of $a_{\ell,f}^{\text{LO-HVP}}$

Combining everything, get $a_{\ell,f}^{\text{LO-HVP}}$ from $C_{TL}^{f}(t)$ [Bernecker et al '11, BMWc '13, Feng et al '13, Lehner '14, ...]

$$a_{\ell,f}^{\text{LO-HVP}}(Q^2 \le Q_{\max}^2) = \lim_{a\to 0,\, L\to\infty,\, T\to\infty} \alpha^2 \left(\frac{a}{m_\ell^2}\right) \sum_{t=0}^{T/2}{}' K(tm_\ell, Q_{\max}^2/m_\ell^2)\ \mathrm{Re}C_{TL}^{f}(t)$$

where

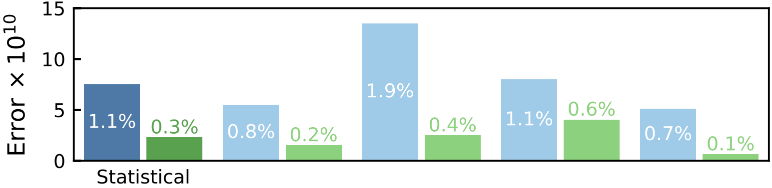
$$K(\tau, r_{\max}) = \int_0^{r_{\max}} dr\, k(r) \left(\tau^2 - \frac{4}{r}\sin^2\frac{\tau\sqrt{r}}{2}\right)$$



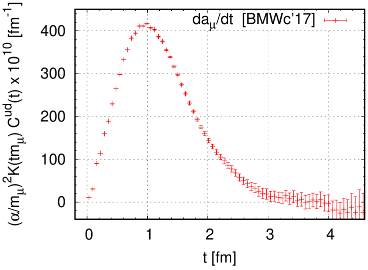

($144 \times 96^3$, $a \sim 0.064$ fm, $M_\pi \sim 135$ MeV)

To be competitive w/ R-ratio, had to reduce total uncertainty $19 \times 10^{-10}$ of BMWc '17 by factor $\sim 4$

# Key improvements: statistical noise reduction




Statistical noise of up and down quark contributions increases exponentially w/ spacetime size of HVP "bubble"

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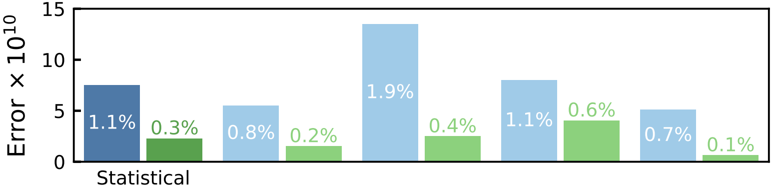
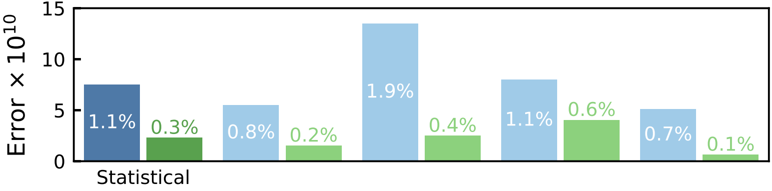



Statistical noise of up and down quark contributions increases exponentially w/ spacetime size of HVP "bubble"

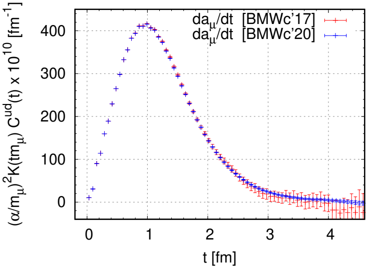
Solve w/:

- Algorithmic improvements (EigCG, solver truncation [Bali et al '09], all mode averaging [Blum et al '13]) to generate more statistics: $> 25,000$ gauge configurations & tens of millions of measurements
- Exact treatment of long-distance modes to reduce long-distance noise (low mode averaging [Neff et al '01, Giusti et al '04, ...])
- Rigorous upper/lower bounds on long-distance contribution [Lehner '16, BMWc '17]

# Key improvements: statistical noise reduction

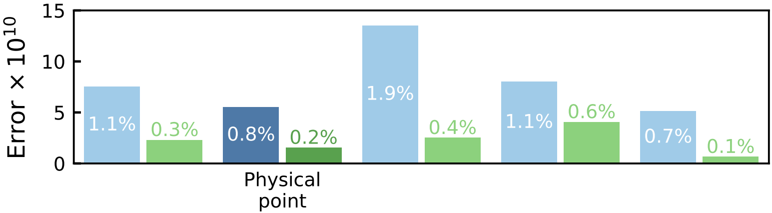



Statistical noise of up and down quark contributions increases exponentially w/ spacetime size of HVP “bubble”




$(144 \times 96^3,\ a \sim 0.064\ \mathrm{fm},\ M_\pi \sim 135\ \mathrm{MeV})$

# Key improvements: tuning of QCD parameters

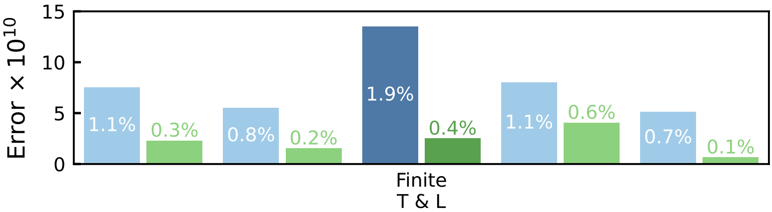



Must tune parameters of QCD very precisely: $m_u$, $m_d$, $m_s$, $m_c$ & overall mass scale

Solve w/:

- Permil determination of overall QCD scale
- Set w/ $\Omega^-$ baryon mass computed w/ 0.2% uncertainty
- Use Wilson flow scale [Lüscher '10, BMWc '12] to separate out electromagnetic corrections

# Key improvements: remove finite spacetime distortions

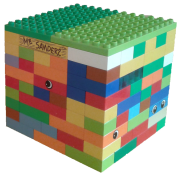



Even on "large" lattices ($L \gtrsim 6\,\mathrm{fm}$, $T \gtrsim 9\,\mathrm{fm}$), early pen-and-paper estimate [Aubin et al '16] suggested that exponentially suppressed finite-volume distortions are still $O(2\%)$

Solve by:

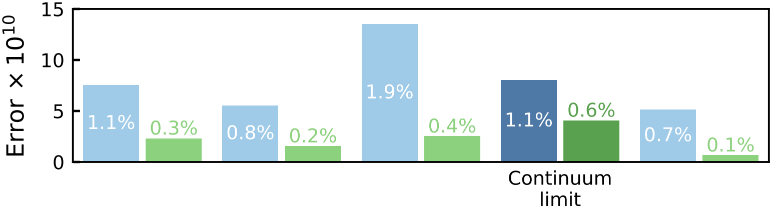
- Finding a way to perform dedicated supercomputer simulations to calculate effect between above and much larger $L = T = 11\,\mathrm{fm}$ volume directly in QCD, i.e. "big" − "ref"
- Computing remnant $\sim 0.1\%$ effect of "big" volume w/ pheno. models of QCD that correctly predict "big" − "ref"

"ref" →

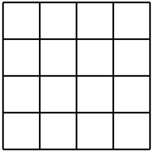



← "big"

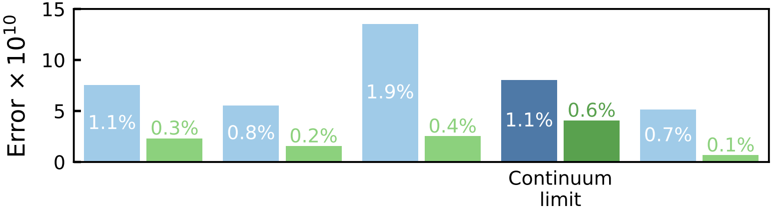
# Key improvements: controlled continuum limit




Our world corresponds to spacetime w/ lattice spacing $a \to 0$



# Key improvements: controlled continuum limit

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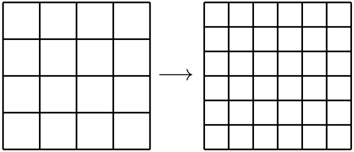

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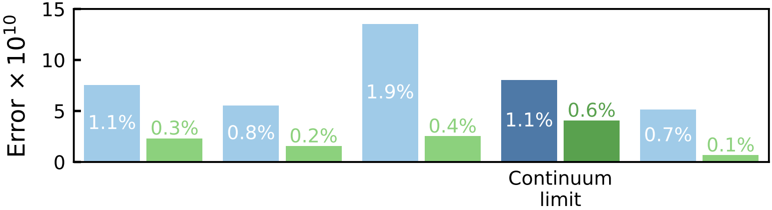


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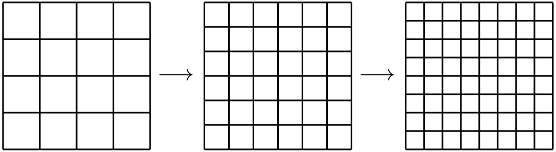

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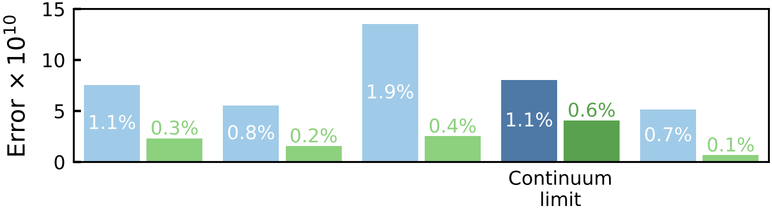


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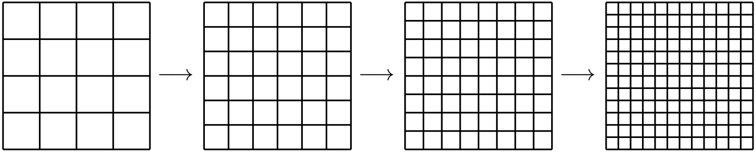

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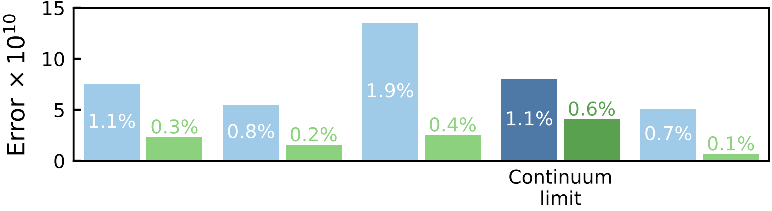


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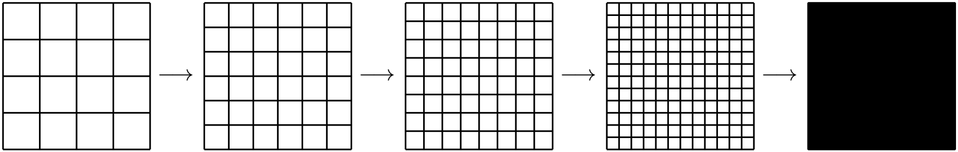

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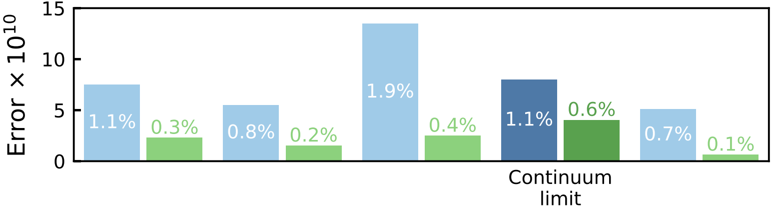
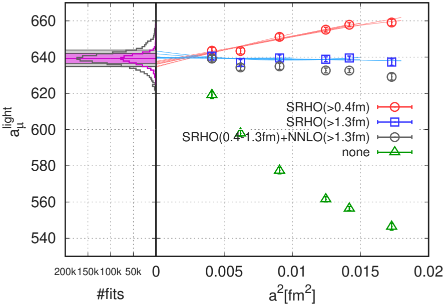


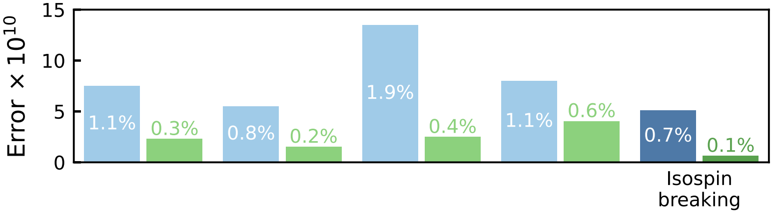

Our world corresponds to spacetime w/ lattice spacing $a \to 0$

Control $a \to 0$ extrapolation of results by:

- Performing all calculations on lattices w/ 6 values of $a$ in range $0.134\,\mathrm{fm} \to 0.064\,\mathrm{fm}$
- Reducing statistical error at smallest $a$ from 1.9% to 0.3% !
- Improving approach to continuum limit w/ pheno. models for QCD [Sakurai '60, Bijnens et al '99, Jegerlehner et al '11, Chakraborty et al '17, BMWc '20] shown to reproduce distortions observed at $a>0$
- Extrapolate results to $a=0$ using theory as guide



# Key improvements: QED and $m_u \neq m_d$ corrections



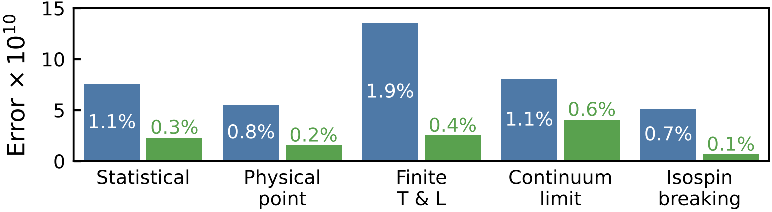

For subpercent accuracy, must include small effects from electromagnetism and due to fact that masses of *u* and *d* quarks are not quite equal

- Effects are proportional to powers of $\alpha = \frac{e^2}{4\pi} \sim 0.01$ and $\frac{m_d - m_u}{(M_p/3)} \sim 0.01$

$\Rightarrow$ for SM calculation at permil accuracy sufficient to take into account contributions proportional to only first power of $\alpha$ or $\frac{m_d - m_u}{(M_p/3)}$

- We include *all* such contributions for *all* calculated quantities needed in calculation

# Robust determination of uncertainties



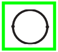

Thorough and robust determination of statistical and systematic uncertainties

- Stat. err.: resampling methods
- Syst. err.: extended frequentist approach [BMWc '08, '14]
  - Hundreds of thousands of different analyses of correlation functions
  - Weighted by AIC weight
  - Use median of distribution for central values & $16 \div 84\%$ confidence interval to get total error

(Nature paper has 95 pp. Supplementary information detailing methods)

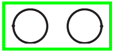
# Summary of contributions to $a_\mu^{\text{LO-HVP}}$

## Isospin symmetric


connected light
633.7(2.1)(4.2)


connected strange
53.393(89)(68)

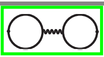
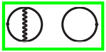

connected charm
14.6(0)(1)


disconnected
-13.36(1.18)(1.36)

## QED isospin-breaking: valence

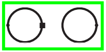


connected -1.23(40)(31)

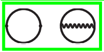


disconnected -0.55(15)(10)

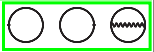
## Strong isospin-breaking


connected
6.60(63)(53)


disconnected
-4.67(54)(69)

## QED isospin-breaking: sea



connected 0.37(21)(24)

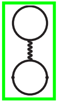


disconnected -0.040(33)(21)

## Etc.

bottom; higher order; perturbative

0.11(4)

## QED isospin-breaking: mixed


connected -0.0093(86)(95)


disconnected 0.011(24)(14)

## Finite-size effects

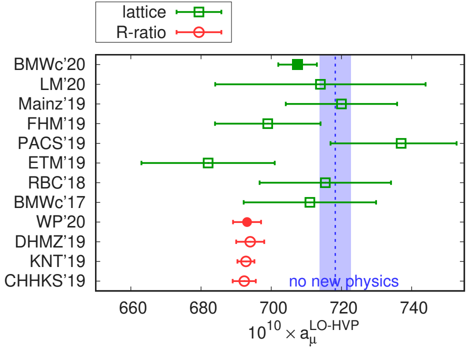
isospin-symmetric
18.7(2.5)

isospin-breaking
0.0(0.1)

$$10^{10}\times a_\mu^{\text{LO-HVP}} = 707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}[5.5]_{\text{tot}}$$

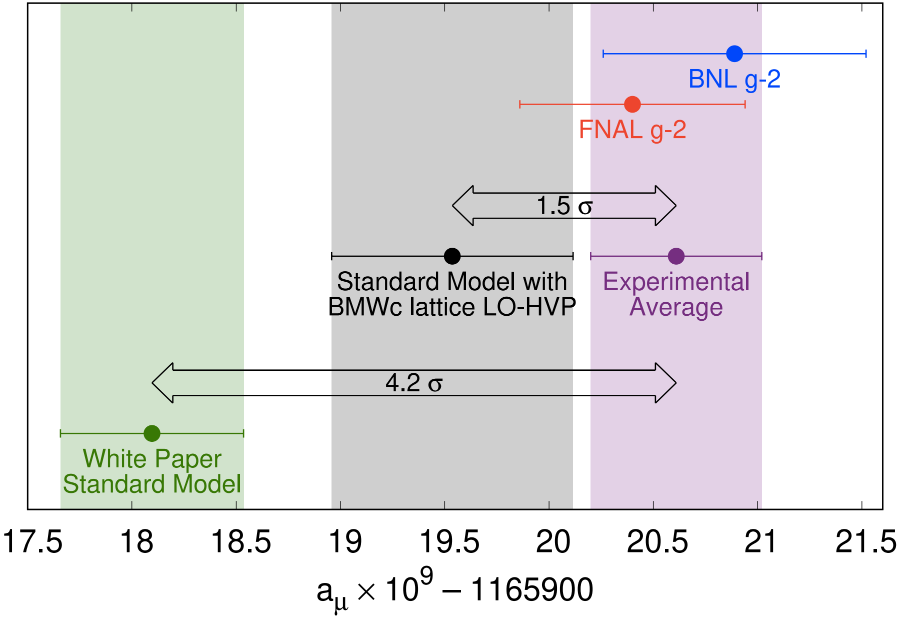
# Comparison and outlook

# Comparison



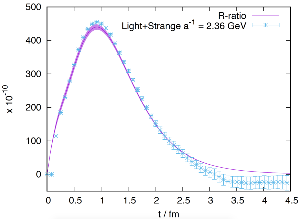

- Consistent with other lattice results
- Total uncertainty is divided by $3 \div 4$ ...
- ... and comparable to R-ratio and experiment
- Consistent w/ experiment @ $1.5\sigma$ ("no new physics" scenario) !
- $2.1\sigma$ larger than R-ratio average value [WP '20]

# Fermilab plot, April 7 2021, BMWc version

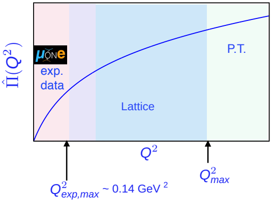


# What next?

- FNAL to reduce its error by factor of $\sim$ 4 in coming years
- HLbL error must be reduced by factor of $1.5 \div 2$
- Must reduce ours by factor of 4 !
- And must reduce proportion of *systematics* in theory error
- Will experiment still agree with our prediction ?
- Must be confirmed by other lattice groups
- If confirmed, must understand why lattice doesn't agree with R-ratio
- If disagreement can be fixed, combine LQCD and phenomenology to improve overall uncertainty [RBC/UKQCD '18]
- Important to pursue $e^+e^- \to$ hadrons measurements [BaBar, CMD-3, BES III, Belle II, . . . ]
- $\mu e \to \mu e$ experiment MUonE very important for experimental crosscheck and complementarity w/ LQCD
- Important to build J-PARC $g_\mu - 2$ and pursue $a_e$ experiments




[RBC/UKQCD '18]




[Marinkovic et al '19]